On the Sliding Block Maxima Method in Extreme Value Statistics

Torben Staud
Defense

Faculty of Mathematics
Ruhr-Universität Bochum
February 3, 2025

On the Sliding Block Maxima Method in Extreme Value Statistics

Torben Staud
Defense

Faculty of Mathematics
Ruhr-Universität Bochum
February 3, 2025

Thesis

- U-statistics of Block Maxima
- Bootstrapping Block Maxima estimators
- Maximal correlation of Marshall Olkin distribution

Thesis

- $\boxed{\text{U-statistics of Block Maxima}}$
- $\boxed{\text{Bootstrapping Block Maxima estimators}}$
- Maximal correlation of Marshall Olkin distribution

Extremes

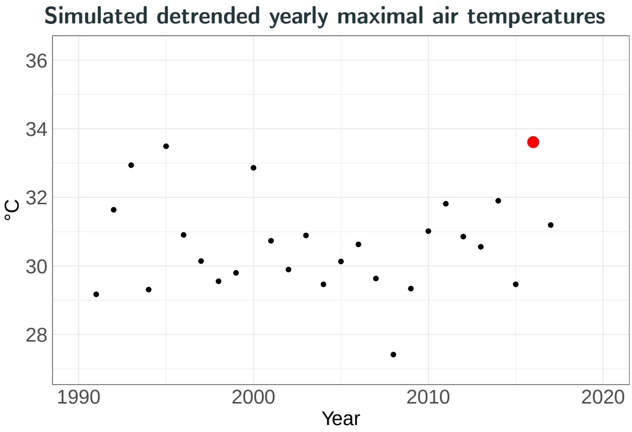
Importance of Extreme Value Statistics

Example

- Parameter of interest: $.99$-quantile of yearly maximal air temperatures.
- Empirical $.99$-quantile estimator degenerates to largest past observation, for “small” sample size.

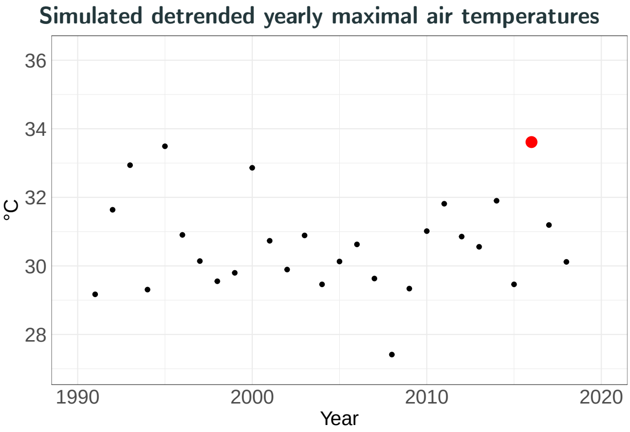
Example

- Parameter of interest: $.99$-quantile of yearly maximal air temperatures.
- Empirical $.99$-quantile estimator degenerates to largest past observation, for "small" sample size.



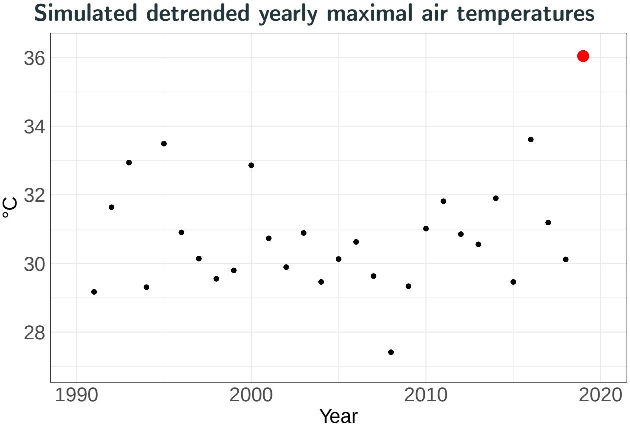
Example

- Parameter of interest: $.99$-quantile of yearly maximal air temperatures.
- Empirical $.99$-quantile estimator degenerates to largest past observation, for "small" sample size.



Example

- Parameter of interest: $.99$-quantile of yearly maximal air temperatures.
- Empirical $.99$-quantile estimator degenerates to largest past observation, for "small" sample size.



Extreme Value Theory/Statistics

Extreme value theory
Asymptotic distribution of *large observations*

Extreme value statistics
Statistical inference for characteristics of this distribution

Extreme value theory
Asymptotic distribution of *large observations*

Extreme value statistics
Statistical inference for characteristics of this distribution

- Block Maxima method
- Peaks Over Threshold method

Extreme Value Theory/Statistics

Extreme value theory
Asymptotic distribution of *large observations*

Extreme value statistics
Statistical inference for characteristics of this distribution

- $\boxed{\text{Block Maxima method}}$
- Peaks Over Threshold method

Why Maxima

- Distributional limit is well understood: Fisher-Tippett-Gnedenko (1928, 1943); Leadbetter (1983)

- Distributional limit is well understood: Fisher-Tippett-Gnedenko (1928, 1943); Leadbetter (1983)

Condition: Let $(X_t)_{t\in\mathbb{N}_0}$ be a time series. There exist (regular) sequences $a_r > 0, b_r \in \mathbb{R}$ and a $\gamma \in \mathbb{R}$ such that

$$\frac{M_{r,1} - b_r}{a_r} \rightsquigarrow \mathrm{GEV}(\gamma), \quad (r \to \infty) \tag{DoA}$$

where $M_{r,1} = \max\{X_1, \ldots, X_r\}$, $\mathrm{GEV}(\gamma) = \mathrm{GEV}(0,1,\gamma)$; γ is called **extreme value index**.

Why Maxima

- Distributional limit is well understood: Fisher-Tippett-Gnedenko (1928, 1943); Leadbetter (1983)

Condition: Let $(X_t)_{t\in\mathbb{N}_0}$ be a time series. There exist (regular) sequences $a_r > 0, b_r \in \mathbb{R}$ and a $\gamma \in \mathbb{R}$ such that

$$\frac{M_{r,1} - b_r}{a_r} \rightsquigarrow \mathrm{GEV}(\gamma), \quad (r \to \infty) \tag{DoA}$$

where $M_{r,1} = \max\{X_1, \dots, X_r\}$, $\mathrm{GEV}(\gamma) = \mathrm{GEV}(0,1,\gamma)$; γ is called **extreme value index**.

Statistical Framework:

- $(X_t)_t$ strictly stationary.
- Short-range dependent.

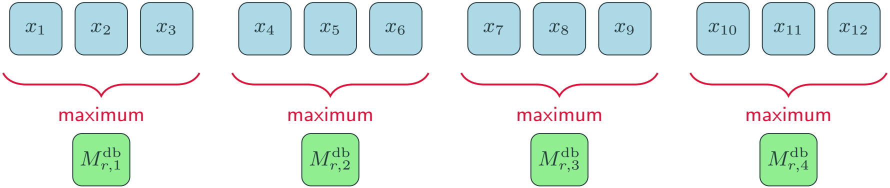
(Disjoint) Block Maxima method I

Observations $x_1, \ldots, x_{12}$; **block size** $r = 3$

x_1 x_2 x_3 | x_4 x_5 x_6 | x_7 x_8 x_9 | x_{10} x_{11} x_{12}

(Disjoint) Block Maxima method I

Observations $x_1, \ldots, x_{12}$; **block size** $r = 3$



Observations $x_1, \ldots, x_{12}$; **block size** $r = 3$

$$\underbrace{x_1 \quad x_2 \quad x_3}_{\substack{\text{maximum} \\ M^{\mathrm{db}}_{r,1}}} \qquad \underbrace{x_4 \quad x_5 \quad x_6}_{\substack{\text{maximum} \\ M^{\mathrm{db}}_{r,2}}} \qquad \underbrace{x_7 \quad x_8 \quad x_9}_{\substack{\text{maximum} \\ M^{\mathrm{db}}_{r,3}}} \qquad \underbrace{x_{10} \quad x_{11} \quad x_{12}}_{\substack{\text{maximum} \\ M^{\mathrm{db}}_{r,4}}}$$

Disjoint Block Maxima sample $\mathcal{M}^{\mathrm{db}}_r = (M^{\mathrm{db}}_{r,1}, \ldots, M^{\mathrm{db}}_{r,4})$

(Disjoint) Block Maxima method II

- Important: Block size $r = r_n \to \infty$, $r = o(n)$.
- **Literature** Gumbel (1958); ...

(Disjoint) Block Maxima method II

- Important: Block size $r = r_n \to \infty$, $r = o(n)$.
- **Literature** Gumbel (1958); ...

Why investigate a new method?

(Disjoint) Block Maxima method II

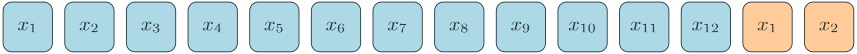
- Important: Block size $r = r_n \to \infty$, $r = o(n)$.
- **Literature** Gumbel (1958); ...

Why investigate a new method?

- Improve (uniformly?) upon estimators

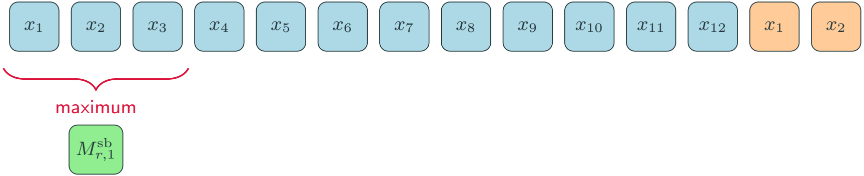
Sliding Block Maxima method I

Observations $x_1, \ldots, x_{12}$; **block size** $r = 3$



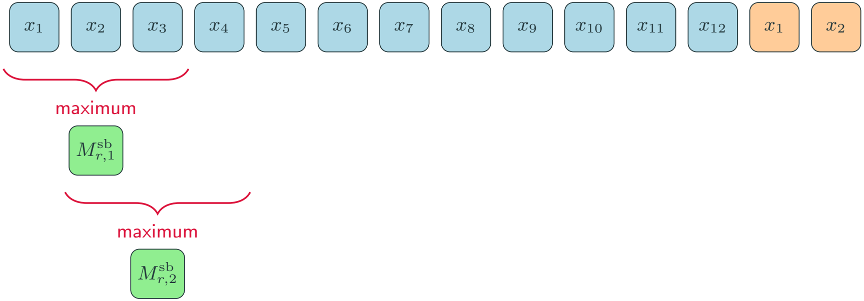
Sliding Block Maxima method I

Observations $x_1, \ldots, x_{12}$; **block size** $r = 3$



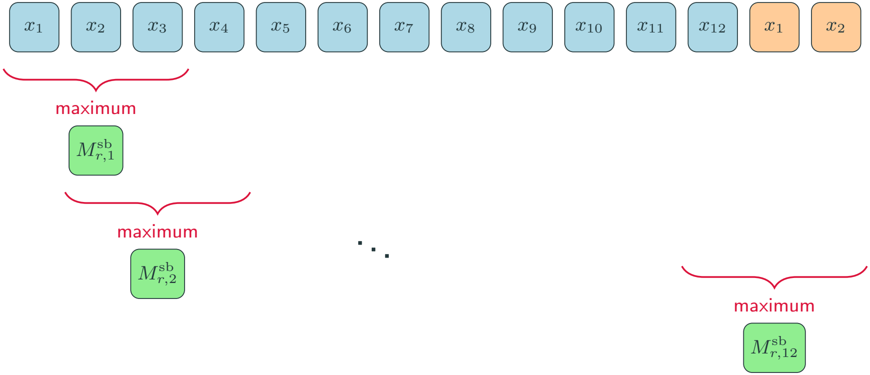
Sliding Block Maxima method I

Observations $x_1, \ldots, x_{12}$; **block size** $r = 3$



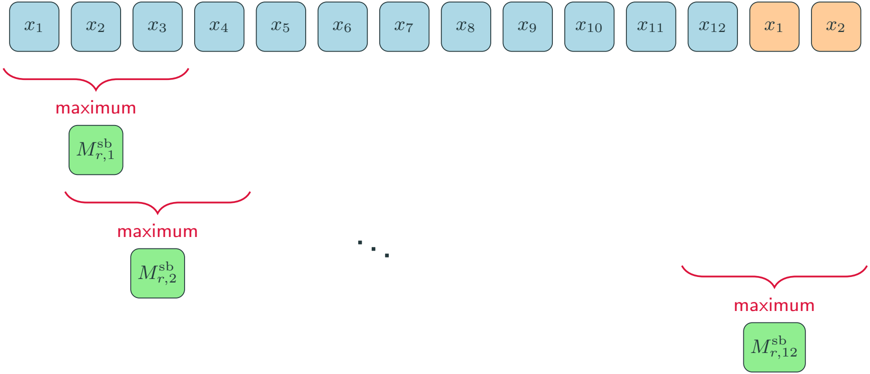
Sliding Block Maxima method I

Observations $x_1, \ldots, x_{12}$; **block size** $r = 3$



Sliding Block Maxima method I

Observations $x_1, \ldots, x_{12}$; **block size** $r = 3$



Sliding Block Maxima sample $\mathcal{M}_r^{\mathrm{sb}} = (M_{r,1}^{\mathrm{sb}}, \ldots, M_{r,12}^{\mathrm{sb}})$

Sliding Block Maxima Method II

- Substantial serial dependence between nearby sliding Block Maxima.
- In literature: Often sliding blocks estimators have **smaller asymptotic variances** than their disjoint counterpart: Bücher and Segers (2018); Bücher and Zanger (2024)

Sliding Block Maxima Method II

- Substantial serial dependence between nearby sliding Block Maxima.
- In literature: Often sliding blocks estimators have **smaller asymptotic variances** than their disjoint counterpart: Bücher and Segers (2018); Bücher and Zanger (2024)

Does latter hold for the following large class of Block Maxima estimators?

U-statistics of Block Maxima

Estimation Problem

- F unknown c.d.f. from a c.d.f.-class $\mathcal{F}$ and for a known $\rho \in \mathbb{N}, h\colon \mathbb{R}^\rho \to \mathbb{R}$ one is interested in:

$$\theta = \mathbb{E}[h(Z_1, \ldots, Z_\rho)],$$

  for $Z_1, \ldots, Z_\rho \sim F$ i.i.d.

Estimation Problem

- F unknown c.d.f. from a c.d.f.-class $\mathcal{F}$ and for a known $\rho \in \mathbb{N}, h\colon \mathbb{R}^\rho \to \mathbb{R}$ one is interested in:

$$\theta = \mathbb{E}[h(Z_1, \ldots, Z_\rho)],$$

  for $Z_1, \ldots, Z_\rho \sim F$ i.i.d.

Examples:

- $\theta = \mathbb{E}[Z_1]$ for $h(x) = x, \rho = 1$.
- $\theta = \mathrm{Var}(Z_1)$ for $h_{\mathrm{Var}}(x, y) = (x - y)^2/2, \rho = 2$.
- Probability weighted moments.

Object of interest

Plug Block Maxima samples into U-statistic:

$$U_n^{\mathrm{db}} := \binom{n/r}{2}^{-1} \sum_{1 \le i < j \le n/r} h(M_{r,i}^{\mathrm{db}}, M_{r,j}^{\mathrm{db}}), \quad U_n^{\mathrm{sb}} := \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} h(M_{r,i}^{\mathrm{sb}}, M_{r,j}^{\mathrm{sb}})$$

to estimate

$$\theta_r = \mathbb{E}[h(M_{r,1}, \widetilde{M}_{r,1})],$$

with $\widetilde{M}_{r,1}$ i.i.d. copy of $M_{r,1}$. **Recall** $M_{r,1} = \max\{X_1, \ldots, X_r\}$.

Object of interest

Plug Block Maxima samples into U-statistic:

$$U_n^{\mathrm{db}} := \binom{n/r}{2}^{-1} \sum_{1 \le i < j \le n/r} h(M_{r,i}^{\mathrm{db}}, M_{r,j}^{\mathrm{db}}), \quad U_n^{\mathrm{sb}} := \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} h(M_{r,i}^{\mathrm{sb}}, M_{r,j}^{\mathrm{sb}})$$

to estimate

$$\theta_r = \mathbb{E}[h(M_{r,1}, \widetilde{M}_{r,1})],$$

with $\widetilde{M}_{r,1}$ i.i.d. copy of $M_{r,1}$. **Recall** $M_{r,1} = \max\{X_1, \ldots, X_r\}$.

Objectives:

- Asymptotics
- Comparison

Asymptotic distribution

Theorem (Bücher and S. 2024): Under conditions, for $(X_n)_n$ satisfying (DoA),

$$\sqrt{n/r} \cdot \frac{U_n^{\mathrm{mb}} - \theta_r}{f(a_r)} \rightsquigarrow \mathcal{N}(0, \sigma^2_{\mathrm{mb}}), \quad (n \to \infty)$$

where f comes from the conditions; σ^2_{mb} depends on h, mb, and γ from (DoA).
Furthermore it holds that $\sigma^2_{\mathrm{db}} \geq \sigma^2_{\mathrm{sb}}$.

Asymptotic distribution

Theorem (Bücher and S. 2024): Under conditions, for $(X_n)_n$ satisfying (DoA),

$$\sqrt{n/r} \cdot \frac{U_n^{\mathrm{mb}} - \theta_r}{f(a_r)} \rightsquigarrow \mathcal{N}(0, \sigma_{\mathrm{mb}}^2), \quad (n \to \infty)$$

where f comes from the conditions; σ_{mb}^2 depends on h, mb, and γ from (DoA).
Furthermore it holds that $\sigma_{\mathrm{db}}^2 \geq \sigma_{\mathrm{sb}}^2$.

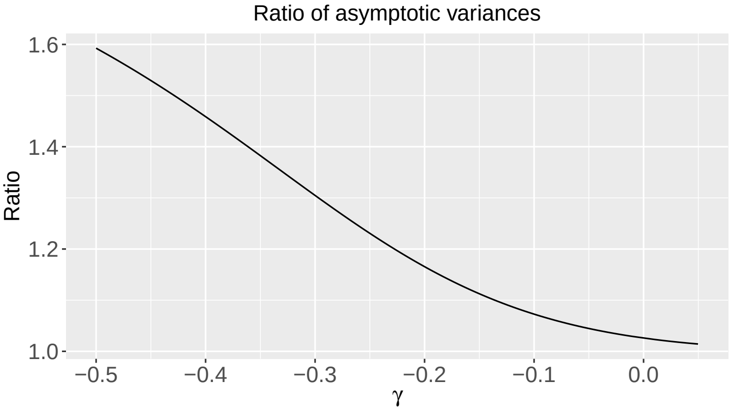
- Often $\sigma_{\mathrm{db}}^2 > \sigma_{\mathrm{sb}}^2$.
- Extendable.

Theoretical comparison for variance estimation

- Estimation of population parameter $\sigma_r^2 = \operatorname{Var}(M_{r,1})$.
- Estimators $U_n^{\mathrm{mb}}(h_{\mathrm{Var}}) = \hat{\sigma}^2_{n,\mathrm{mb}}; \mathrm{mb} \in \{\mathrm{db}, \mathrm{sb}\}$.
- $\lim_{n\to\infty} \operatorname{Var}(\hat{\sigma}^2_{n,\mathrm{db}}) / \operatorname{Var}(\hat{\sigma}^2_{n,\mathrm{sb}})$.
- **Recall**: $\mathcal{L}(M_{r,1}) \approx \mathrm{GEV}(b_r, a_r, \gamma)$ by (DoA).

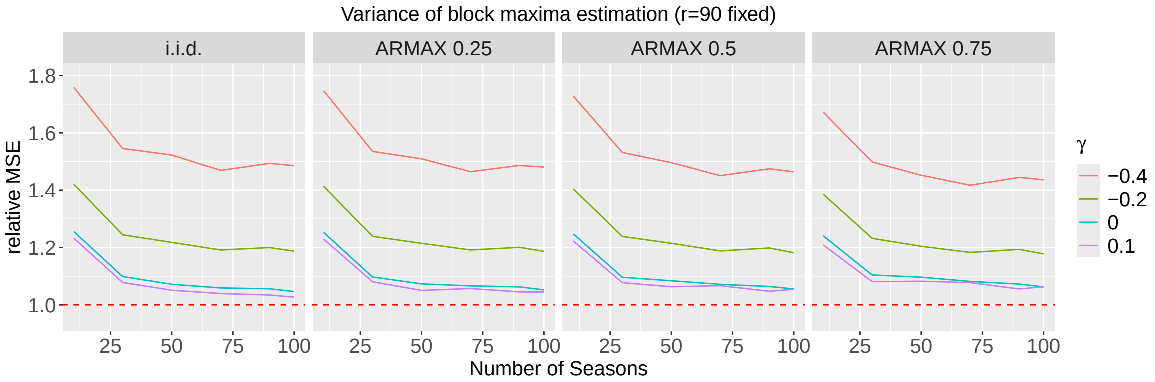
Theoretical comparison for variance estimation

- Estimation of population parameter $\sigma_r^2 = \operatorname{Var}(M_{r,1})$.
- Estimators $U_n^{\mathrm{mb}}(h_{\mathrm{Var}}) = \hat{\sigma}^2_{n,\mathrm{mb}}; \mathrm{mb} \in \{\mathrm{db}, \mathrm{sb}\}$.
- $\lim_{n\to\infty} \operatorname{Var}(\hat{\sigma}^2_{n,\mathrm{db}}) / \operatorname{Var}(\hat{\sigma}^2_{n,\mathrm{sb}})$.
- **Recall**: $\mathcal{L}(M_{r,1}) \approx \mathrm{GEV}(b_r, a_r, \gamma)$ by (DoA).



Empirical comparison for variance estimation

- $\mathrm{MSE}(\hat{\sigma}^2_{n,\mathrm{db}})/\,\mathrm{MSE}(\hat{\sigma}^2_{n,\mathrm{sb}})$ in max-autoregressive time series models.
- Based on $N = 10^4$ repetitions.
- $(X_t)_t$ has $\mathrm{GPD}(\gamma)$ marginals.



Bootstrapping Block Maxima Estimators

Bootstrapping Block Maxima: I

Variance of Block Maxima based estimators[1] might look like

[1] Here: asymptotic variance of $U_n^{(\mathrm{sb})}(h_{\mathrm{Var}})$

Variance of Block Maxima based estimators[1] might look like

$$\sigma^2_{\mathrm{sb}} = \begin{cases} \frac{2}{3\gamma^3}\left(-3g_4 I_{2,2} + 8g_1 g_3 I_{2,1} - 6g_1^2 g_2 I_{1,1}\right), & \gamma > 0 \\ \frac{8}{\gamma^2}\left(\Gamma(-4\gamma) I_{2,2} - 2g_1\Gamma(-3\gamma) I_{2,1} + g_1^2\Gamma(-2\gamma) I_{1,1}\right), & \gamma < 0 \\ 2\zeta(3) - 48 - \frac{8}{3}\pi^2 + \frac{32}{3}\log^3(2) - 48\log^2(2) + 96\log(2) + \frac{16}{3}\pi^2\log(2), & \gamma = 0 \end{cases},$$

where $g_j := \Gamma(1 - j\gamma), j < 1/\gamma$;

$$I_{i,k} := \int_0^{1/2} \left(\alpha_{(j+k)\gamma}(w) - 1\right)\left\{w^{-j\gamma-1}(1-w)^{-k\gamma-1} + w^{-k\gamma-1}(1-w)^{-j\gamma-1}\right\} \mathrm{d}w$$

and

$$\alpha_\beta \colon (0,1) \to (0,\infty), \quad w \mapsto \alpha_\beta(w) = \begin{cases} \frac{1-(1-w)^{\beta+1}}{w(\beta+1)}, & \beta \neq -1 \\ -\frac{\log(1-w)}{w}, & \beta = -1 \end{cases}.$$

[1] Here: asymptotic variance of $U_n^{(\mathrm{sb})}(h_{\mathrm{Var}})$

Bootstrapping Block Maxima: II

Literature:

- Surprisingly: Little research on formal bootstrap consistency results of Block Maxima estimators.
- Recent notable exception: de Haan and Zhou (2024).

Bootstrapping Block Maxima: II

Literature:

- Surprisingly: Little research on formal bootstrap consistency results of Block Maxima estimators.
- Recent notable exception: de Haan and Zhou (2024).

Objective:

Consistent bootstrap methods

Statistical setting

Estimation of $\theta_r = \mathbb{E}[h(M_{r,1})]$ where $h\colon \mathbb{R} \to \mathbb{R}$ satisfies minimal regularity conditions.

- Obtain the following sensible estimators for θ_r.

Statistical setting

Estimation of $\theta_r = \mathbb{E}[h(M_{r,1})]$ where $h\colon \mathbb{R} \to \mathbb{R}$ satisfies minimal regularity conditions.

- Obtain the following sensible estimators for θ_r.

$$\hat{\theta}_n^{\mathrm{mb}} = \frac{1}{n_{\mathrm{mb}}} \sum_{i=1}^{n_{\mathrm{mb}}} h\left(M_{r,i}^{\mathrm{mb}}\right), \quad \mathrm{mb} \in \{\mathrm{db}, \mathrm{sb}\}.$$

where $n_{\mathrm{db}} = n/r, n_{\mathrm{sb}} = n$.

Statistical setting

Estimation of $\theta_r = \mathbb{E}[h(M_{r,1})]$ where $h\colon \mathbb{R} \to \mathbb{R}$ satisfies minimal regularity conditions.

- Obtain the following sensible estimators for θ_r.

$$\hat{\theta}_n^{\mathrm{mb}} = \frac{1}{n_{\mathrm{mb}}} \sum_{i=1}^{n_{\mathrm{mb}}} h\Big(M_{r,i}^{\mathrm{mb}}\Big), \quad \mathrm{mb} \in \{\mathrm{db}, \mathrm{sb}\}.$$

where $n_{\mathrm{db}} = n/r, n_{\mathrm{sb}} = n$.

- "comprises": Moment estimators, PWM estimator, Fréchet (Pseudo) MLE.

Statistical setting

Estimation of $\theta_r = \mathbb{E}[h(M_{r,1})]$ where $h\colon \mathbb{R} \to \mathbb{R}$ satisfies minimal regularity conditions.

- Obtain the following sensible estimators for θ_r.

$$\hat\theta_n^{\mathrm{mb}} = \frac{1}{n_{\mathrm{mb}}} \sum_{i=1}^{n_{\mathrm{mb}}} h\Big(M_{r,i}^{\mathrm{mb}}\Big), \quad \mathrm{mb} \in \{\mathrm{db}, \mathrm{sb}\}.$$

where $n_{\mathrm{db}} = n/r, n_{\mathrm{sb}} = n$.

- "comprises": Moment estimators, PWM estimator, Fréchet (Pseudo) MLE.

Aim: Bootstrap $\hat\theta_n^{\mathrm{mb}} - \theta_r$

Disjoint Block Maxima bootstrapping

Motivation:

- Recall: No overlap between disjoint Block Maxima.
- Treat $\mathcal{M}_r^{\mathrm{db}}$ as i.i.d. sample.
- Use Efron's bootstrap on $\mathcal{M}_r^{\mathrm{db}}$, denoted by $\hat{\theta}_n^{\mathrm{db},*}$.

Disjoint Block Maxima bootstrapping

Motivation:

- Recall: No overlap between disjoint Block Maxima.
- Treat $\mathcal{M}_r^{\mathrm{db}}$ as i.i.d. sample.
- Use Efron's bootstrap on $\mathcal{M}_r^{\mathrm{db}}$, denoted by $\hat{\theta}_n^{\mathrm{db},*}$.

Theorem (Bücher and S. 2024): Under conditions, for $(X_n)_n$ satisfying (DoA), the bootstrap $\hat{\theta}^{\mathrm{db},*}$ is consistent for $\hat{\theta}_n^{\mathrm{db}}$, that is:

$$d_K\Big(\mathcal{L}\big(\hat{\theta}_n^{\mathrm{db},*} - \hat{\theta}_n^{\mathrm{db}} \mid X_1, \dots, X_n\big), \mathcal{L}\big(\hat{\theta}_n^{\mathrm{db}} - \theta_r\big)\Big) = o_{\mathbb{P}}(1).$$

Motivation:

- Naive block bootstrap applied to $\mathcal{M}_r^{\mathrm{sb}}$ is inconsistent.
- Blocks we draw should be asymptotically independent.
- Want to keep the advantage of smaller variance.

Sliding Block Maxima bootstrapping

Motivation:

- Naive block bootstrap applied to $\mathcal{M}_r^{\mathrm{sb}}$ is inconsistent.
- Blocks we draw should be asymptotically independent.
- Want to keep the advantage of smaller variance.

$\rightarrow$ Need a new way of constructing Block Maxima.

Circular Block Maxima

Observations $x_1, \ldots, x_{12}$; **block size** $r = 3$

$\boxed{x_1}\ \boxed{x_2}\ \boxed{x_3} \qquad \boxed{x_4}\ \boxed{x_5}\ \boxed{x_6} \qquad \boxed{x_7}\ \boxed{x_8}\ \boxed{x_9} \qquad \boxed{x_{10}}\ \boxed{x_{11}}\ \boxed{x_{12}}$

Circular Block Maxima

Observations $x_1, \ldots, x_{12}$; **block size** $r = 3$

$x_1 \quad x_2 \quad x_3 \qquad x_4 \quad x_5 \quad x_6 \qquad x_7 \quad x_8 \quad x_9 \qquad x_{10} \quad x_{11} \quad x_{12}$

Divide observations into disjoint blocks of $2 * r = 6$

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \qquad x_7 \quad x_8 \quad x_9 \quad x_{10} \quad x_{11} \quad x_{12}$

Circular Block Maxima

Observations $x_1, \dots, x_{12}$; **block size** $r = 3$

$x_1 \; x_2 \; x_3 \qquad x_4 \; x_5 \; x_6 \qquad x_7 \; x_8 \; x_9 \qquad x_{10} \; x_{11} \; x_{12}$

Divide observations into disjoint blocks of $2 * r = 6$

$x_1 \; x_2 \; x_3 \; x_4 \; x_5 \; x_6 \qquad x_7 \; x_8 \; x_9 \; x_{10} \; x_{11} \; x_{12}$

Repeat first two $(= r - 1)$ observations of each block of blocks at the end

$x_1 \; x_2 \; x_3 \; x_4 \; x_5 \; x_6 \; x_1 \; x_2 \qquad x_7 \; x_8 \; x_9 \; x_{10} \; x_{11} \; x_{12} \; x_7 \; x_8$

Circular Block Maxima

Observations $x_1, \ldots, x_{12}$; **block size** $r = 3$

$x_1 \quad x_2 \quad x_3 \qquad x_4 \quad x_5 \quad x_6 \qquad x_7 \quad x_8 \quad x_9 \qquad x_{10} \quad x_{11} \quad x_{12}$

Divide observations into disjoint blocks of $2 * r = 6$

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \qquad x_7 \quad x_8 \quad x_9 \quad x_{10} \quad x_{11} \quad x_{12}$

Repeat first two $(= r - 1)$ observations of each block of blocks at the end

$$\underbrace{x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_1 \quad x_2}_{\text{sliding maximum}} \qquad \underbrace{x_7 \quad x_8 \quad x_9 \quad x_{10} \quad x_{11} \quad x_{12} \quad x_7 \quad x_8}_{\text{sliding maximum}}$$

$M_1^{\text{cb}} \quad M_2^{\text{cb}} \quad M_3^{\text{cb}} \quad M_4^{\text{cb}} \quad M_5^{\text{cb}} \quad M_6^{\text{cb}} \qquad M_7^{\text{cb}} \quad M_8^{\text{cb}} \quad M_9^{\text{cb}} \quad M_{10}^{\text{cb}} \quad M_{11}^{\text{cb}} \quad M_{12}^{\text{cb}}$

Circular Block Maxima

Observations $x_1, \dots, x_{12}$; block size $r = 3$

$x_1 \quad x_2 \quad x_3 \qquad x_4 \quad x_5 \quad x_6 \qquad x_7 \quad x_8 \quad x_9 \qquad x_{10} \quad x_{11} \quad x_{12}$

Divide observations into disjoint blocks of $2 * r = 6$

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \qquad x_7 \quad x_8 \quad x_9 \quad x_{10} \quad x_{11} \quad x_{12}$

Repeat first two $(= r - 1)$ observations of each block of blocks at the end

$$\underbrace{x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_1 \quad x_2}_{\text{sliding maximum}} \qquad \underbrace{x_7 \quad x_8 \quad x_9 \quad x_{10} \quad x_{11} \quad x_{12} \quad x_7 \quad x_8}_{\text{sliding maximum}}$$

$M_1^{\mathrm{cb}} \quad M_2^{\mathrm{cb}} \quad M_3^{\mathrm{cb}} \quad M_4^{\mathrm{cb}} \quad M_5^{\mathrm{cb}} \quad M_6^{\mathrm{cb}} \qquad M_7^{\mathrm{cb}} \quad M_8^{\mathrm{cb}} \quad M_9^{\mathrm{cb}} \quad M_{10}^{\mathrm{cb}} \quad M_{11}^{\mathrm{cb}} \quad M_{12}^{\mathrm{cb}}$

Circular Block Maxima sample $\mathcal{M}_r^{\mathrm{cb}} = (M_{r,1}^{\mathrm{cb}}, \dots, M_{r,12}^{\mathrm{cb}})$

Sliding Block Maxima bootstrapping continued

Use Efron's[2] bootstrap on $\mathcal{M}_r^{(\mathrm{cb})}$ denoted by $\hat{\theta}^{(\mathrm{cb}),*}$

Theorem (Bücher and S. 2024): Under conditions, for $(X_n)_n$ satisfying (DoA), the bootstrap $\hat{\theta}^{\mathrm{cb},*}$ is consistent for $\hat{\theta}_n^{\mathrm{sb}}$, that is:

$$d_K\Big(\mathcal{L}\big(\hat{\theta}_n^{\mathrm{cb},*} - \hat{\theta}_n^{\mathrm{cb}} \mid X_1, \ldots, X_n\big), \mathcal{L}\big(\hat{\theta}_n^{\mathrm{mb}} - \theta_r\big)\Big) = o_{\mathbb{P}}(1),$$

$\mathrm{mb} \in \{\mathrm{sb}, \mathrm{cb}\}$.

[2] Or multiplier bootstraps

Sliding Block Maxima bootstrapping continued

Use Efron's[2] bootstrap on $\mathcal{M}_r^{(\mathrm{cb})}$ denoted by $\hat{\theta}^{(\mathrm{cb}),*}$

Theorem (Bücher and S. 2024): Under conditions, for $(X_n)_n$ satisfying (DoA), the bootstrap $\hat{\theta}^{\mathrm{cb},*}$ is consistent for $\hat{\theta}_n^{\mathrm{sb}}$, that is:

$$d_K\Big(\mathcal{L}\big(\hat{\theta}_n^{\mathrm{cb},*} - \hat{\theta}_n^{\mathrm{cb}} \mid X_1, \dots, X_n\big), \mathcal{L}\big(\hat{\theta}_n^{\mathrm{mb}} - \theta_r\big)\Big) = o_{\mathbb{P}}(1),$$

$\mathrm{mb} \in \{\mathrm{sb}, \mathrm{cb}\}$.

- $\hat{\theta}_n^{\mathrm{cb},*}$ consistent for both cb and sb.

[2] Or multiplier bootstraps

Conclusion

- Sliding method preferable over disjoint in U-statistic framework.
- Both theoretically as in simulation studies.
- Extendable: E.g. to a non-stationary suited for environmental applications.

Conclusion

- Sliding method preferable over disjoint in U-statistic framework.
- Both theoretically as in simulation studies.
- Extendable: E.g. to a non-stationary suited for environmental applications.
- Presented novel approach of constructing Block Maxima.
- Lead to formal bootstrap consistency results for sliding estimators.

References

- Bücher, A. and Segers, J. (2018). Inference for heavy tailed stationary time series based on sliding blocks. Electron. J. Stat., 12(1):1098–1125.
- Bücher, A. and Zanger, L. (2023). On the disjoint and sliding Block Maxima method for piecewise stationary time series. Ann. Statist., 51(2):573–598.
- de Haan, L. and Zhou, C. (2024). Bootstrapping extreme value estimators. Journal of the American Statistical Association, 119(545):382–393.
- Fisher RA, Tippett LHC (1928). Limiting forms of the frequency distribution of the largest or smallest member of a sample. Mathematical Proceedings of the Cambridge Philosophical Society. 24(2):180-190.
- Gumbel, E. J. (1958). Statistics of extremes. Columbia University Press, New York.
- Leadbetter, M. R. (1983). Extremes and local dependence in stationary sequences. Z. Wahrsch. Verw. Gebiete, 65(2):291–306
- Yoshihara, K.-i. (1976). Limiting behavior of U-statistics for stationary, absolutely regular processes. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 35(3):237–252.

Papers

Bücher, A. and S., T. (2024). Limit theorems for non-degenerate U-statistics of block maxima for time series. Electron. J. Statist., 18(2):2850–2885.

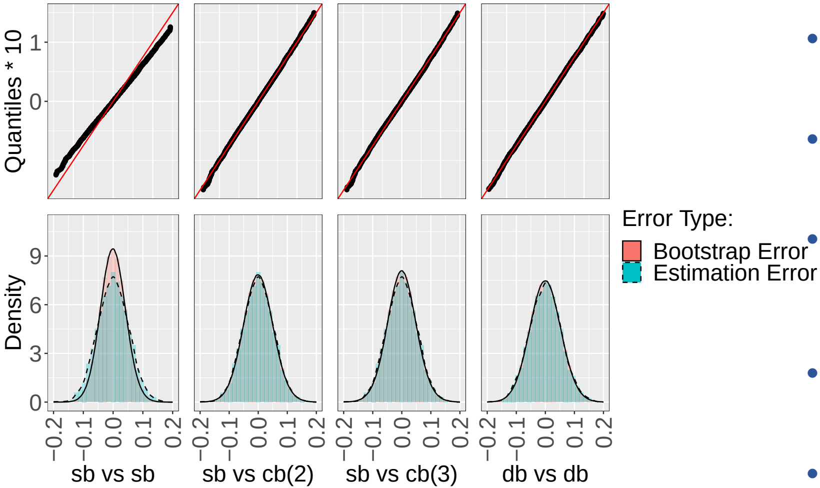
Bücher, A. and S., T. (2024+). Bootstrapping estimators based on the Block Maxima method. Under revision at *Journal of the Royal Statistical Society Series B: Statistical Methodology*

Bücher, A. and S., T. (2025). On the maximal correlation coefficient for the bivariate Marshall Olkin distribution. *Statistics and Probability Letters*, Vol. 219, 110323.

Thank you

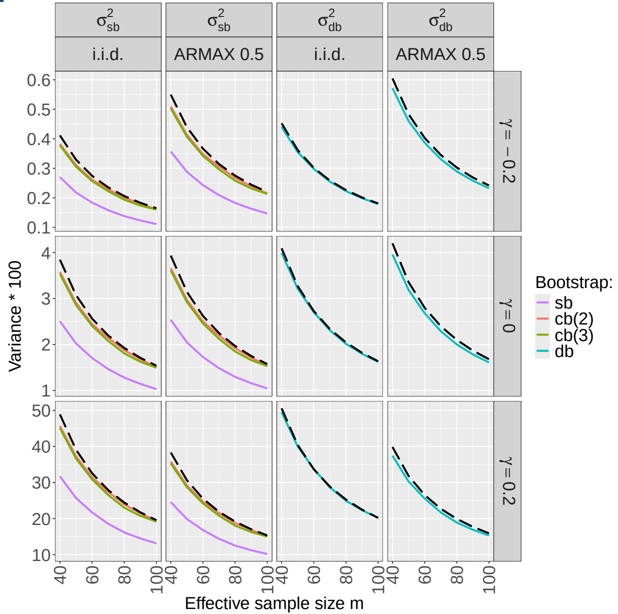
Appendix

Bootstrap simulation study: Mean of a block maximum I



- Estimate mean of a block maximum of size $r = 90$: μ_{90}
- Estimator: Arithmetic mean of block maxima sample: $\hat{\mu}_90$.
- Model: ARMAX(0.5) with $\mathrm{GPD}(-0.2)$ marginals, $m = 80$ seasons, B = 1,000 bootstrap replications.
- Naive sliding bootstrap is inconsistent (variance is smaller).
- cb and db are consistent.

Bootstrap simulation study: Mean of a block maximum II



- Estimate $\mathrm{Var}(\hat{\mu}_{90})$ with Block Maxima bootstrap and a fixed block size $r = 90$.
- Model: ARMAX(0.5) with $\mathrm{GPD}(\gamma)$ marginals, and varying season numbers; B $=$ 1,000 bootstrap replications.
- Naive sliding variance estimator underestimates the true variance.

Theorem (Fisher-Tippett-Gnedenko, 1928-1943): Suppose for i.i.d.[3] r.v.s $X_i \sim F$, there are normalizing sequences $a_n > 0, b_n \in \mathbb{R}$ and a non-degenerate limiting distribution G satisfying

$$\frac{M_n - b_n}{a_n} \rightsquigarrow G,$$

where $M_n := \max(X_1, \ldots, X_n)$. Then $G \sim \mathrm{GEV}(\mu, \sigma, \gamma)$ for a shape parameter $\gamma \in \mathbb{R}$ depending on F and location, scale parameters $\mu, \sigma \in \mathbb{R} \times (0, \infty)$.

Note: Surprisingly: Class of GEV is *small*
Heuristical explanation: Since $\mathbb{P}(M_n \leq x) = F^n(x)$ for $x \in \mathbb{R}$ one can show

$$G(x) = G^k(c_k x + d_k) \quad \text{(Max Stability of } G\text{)}$$

for $k \in \mathbb{N}$ and some $c_k > 0, d_k \in \mathbb{R}$.

[3] Extendable to stationary time series satisfying the $D(u_n)$ condition (Leadbetter, 1983)

GEV distribution I

Some facts on the $\mathrm{GEV}(\eta)$ distribution:

Let $\eta = (\mu, \sigma, \gamma) \in \mathbb{R} \times (0, \infty) \times \mathbb{R}$.

- **c.d.f.**

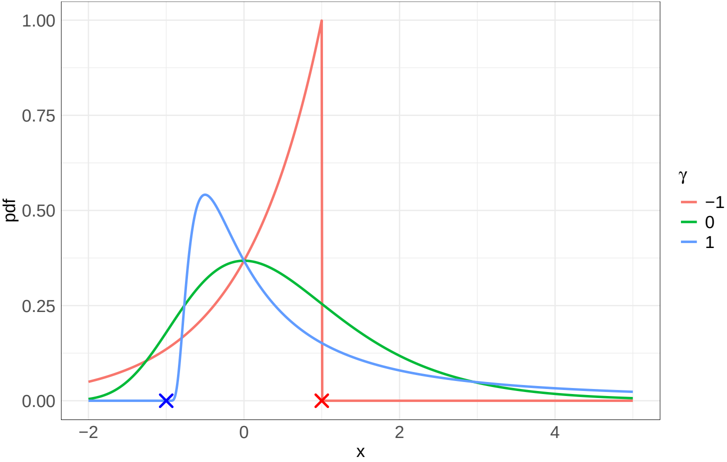
$$G(x) = \exp\big[-(1+\gamma z)^{-1/\gamma}\big]\mathbb{1}(1+\gamma z > 0).$$

- **Support**: Depends on η.
- **Moments**: $\mathbb{E}[|Z|^k] < \infty$ iff $1/k < \gamma$.
- **Parameterization**: Corresponds to the Reverse-Weibull family $(\gamma < 0)$, Gumbel family $(\gamma = 0)$, Fréchet family $(\gamma > 0)$.

GEV distribution II

p.d.f

$$g(x) = \frac{1}{\sigma}(1+\gamma z)^{-(1/\gamma+1)} G(x) \mathbb{1}(1+\gamma z > 0)$$



Kernel transformation condition

Problem:

Only the rescaled Block Maxima $Z_{r,i}^{\mathrm{mb}} := (M_{r,i}^{\mathrm{mb}} - b_r)/a_r$ have distributional limits and in general $h(M_{r,i}^{\mathrm{mb}}, M_{r,j}^{\mathrm{mb}}) \neq h(Z_{r,i}^{\mathrm{mb}}, Z_{r,j}^{\mathrm{mb}})$.

Solution:

Suppose h satisfies the following kernel transformation condition:

(Simplified) Kernel condition: There exist functions $f\colon (0,\infty)\times\mathbb{R}\to(0,\infty), \ell\colon (0,\infty)\times\mathbb{R}\to\mathbb{R}$ such that for $b\in\mathbb{R}, a>0, x, y\in\mathbb{R}$

$$h\left(\frac{x-b}{a}, \frac{y-b}{a}\right) = \frac{h(x,y)}{f(a,b)} + \ell(a,b). \tag{KT}$$

Examples:

- variance kernel, PWM, Kendall's τ, covariance

Conditions for U-statistics

- Block size: $r = r_n \to \infty, r = o(n)$.
- Intermediate sequence: $\ell_n = o(r_n), nr^{-1}\alpha(\ell) = o(1), r\ell^{-1}\alpha(\ell) = o(1)$.
- Short range dependence: $(n/r)^{1+\omega}\beta(r) = o(1)$, for an $\omega > 0$.
- Asymptotic integrability of h w.r.t. $M_{r,i}, M_{r,j}, \widetilde{M}_{r,i}$, that is

$$\limsup_{r} \int\int |h(x,y)|^{2+\nu} \, \mathrm{d}P_{Z_{r,1}}^{\otimes 2}(x,y) < \infty, \quad \limsup_{n} \sup_{s} \int |h(x,y)|^{2+\nu} \, \mathrm{d}P_{(Z_{r,1}, Z_{r,1+s})}(x,y) < \infty.$$

- Regularity $h \colon \mathbb{R}^2 \to \bar{\mathbb{R}}$ is λ^2-a.e. continuous, bounded on bounded sets and satisfies (KT).

Form of asymptotic variance

- Let $Z \sim \mathrm{GEV}(\gamma)$ with c.d.f. G.
- Let $(Z_{1,\xi}, Z_{2,\xi})$ have the c.d.f. $G_\xi(x,y) = G^\xi(x)G^\xi(y)G^{1-\xi}(x \wedge y), \xi \in [0,1]$.
- Let $h_1(x) := \mathbb{E}[h(x,Z)]$.

$$\sigma^2_{\mathrm{db}} = 4\,\mathrm{Var}(h_1(Z)), \quad \sigma^2_{\mathrm{sb}} = 8\int_0^1 \mathrm{Cov}\left[h_1(Z_{1,\xi}), h_1(Z_{2,\xi})\right] \mathrm{d}\xi.$$

Why does $\sigma^2_{\mathrm{db}} \geq \sigma^2_{\mathrm{sb}}$ hold?

$$\begin{aligned}\sigma^2_{\mathrm{sb}} &= 8\,\mathrm{Var}(h_1(Z)) \cdot \int_0^1 \mathrm{Cor}\left[h_1(Z_{1,\xi}), h_1(Z_{2,\xi})\right] \mathrm{d}\xi \\ &\leq 8\,\mathrm{Var}(h_1(Z)) \cdot \int_0^1 (1-\xi)\,\mathrm{d}\xi = \sigma^2_{\mathrm{db}}.\end{aligned}$$

U-statistics asymptotic normality: Proof sketch

1. By kernel transformation condition: Enough to consider $Z_{r,i} := (M_{r,i} - b_r)/a_r$. $U^{\mathrm{mb}}_{n,Z}$ denotes the U-statistic with Z_r^{mb} instead of M_r^{mb}. Supress mb in the following.
2. $\vartheta_r := \mathbb{E}[h(Z_{r,1}, \widetilde{Z}_{r,1})]$.
3. Use subasymptotic Hoeffding-decomposition:
$$U^{\mathrm{mb}}_{n,Z} - \vartheta_r = \frac{2}{n_{\mathrm{mb}}} \sum_{i=1}^{n_{\mathrm{mb}}} h_{1,r}(Z_{r,i}) + \frac{2}{n_{\mathrm{mb}}(n_{\mathrm{mb}} - 1)} \sum_{1 \le i < j \le n_{\mathrm{mb}}} h_{2,r}(Z_{r,i}, Z_{r,j}) = L_n + D_n,$$
where $h_{1,r}(x) := \mathbb{E}[h(x, Z_r)] - \vartheta_r$, $h_{2,r}(x,y) = h(x,y) - h_{1,r}(x) - h_{1,r}(y) - \vartheta_r$.
4. Show $\sqrt{n/r} L_n \rightsquigarrow \mathcal{N}(0, \sigma^2_{\mathrm{mb}})$.
5. Show $\sqrt{n/r} D_n \rightsquigarrow 0$.

Proof continued

4.: $\sqrt{n/r}L_n \rightsquigarrow \mathcal{N}(0, \sigma^2_{\mathrm{mb}})$

1. Show
$$h_{1,r}(Z_{r,1}) \rightsquigarrow h_1(Z);$$
use Wichura's theorem.
2. Show convergence of variances. For (db): May assume $(Z_{r,i})$ are i.i.d. For (sb): Bernstein blocking, identification as integral.

5.: $\sqrt{n/r}D_n \rightsquigarrow 0$.

1. Show L^2 convergence to null.
2. Berbee coupling with smallest/biggest index random variable.
3. Condition on $(Z_{r,i_2}, Z_{r,j_1}, Z_{r,j_2})$.

Joint convergence of sliding block maxima

1. Reason why sliding variance is smaller than disjoint variance.
2. Write $Z_{r,i}$ for $Z_{r,i}^{\mathrm{sb}}$. Let $\xi \in (0,1)$ (otherwise asymptotically independent)

$$
\begin{aligned}
&\mathbb{P}(Z_{r,1} \le x, Z_{r,1+\lfloor \xi r \rfloor} \le y) \\
&= \mathbb{P}\left(X_1, \dots X_{\lfloor \xi r \rfloor} \le a_r x + b_r, X_{1+\lfloor \xi r \rfloor}, \dots, X_r \le a_r(x \wedge y) + b_r, X_{r+1}, \dots, X_{r+\lfloor \xi r \rfloor} \le a_r y + b_r\right) \\
&= G\left[\xi^{-\gamma} x + \frac{\xi^{-\gamma} - 1}{\gamma}\right] \cdot G\left[(1-\xi)^{-\gamma}(x \wedge y) + \frac{(1-\xi)^{-\gamma} - 1}{\gamma}\right] \cdot G\left[\xi^{-\gamma} y + \frac{\xi^{-\gamma} - 1}{\gamma}\right] + o_{\mathbb{P}}(1) \\
&\to G^{\xi}(x) G^{\xi}(y) G^{1-\xi}(x \wedge y) = G_{\xi}(x,y).
\end{aligned}
$$

- Obvious: G_ξ is a bivariate EVD.
- G_ξ has EVC $C_{1-\xi,1-\xi}$ Marshall-Olkin copula with parameter $\varphi = 1-\xi, \psi = 1-\xi$.

Asymptotic normality with bias (U-statistics)

Theorem: Under conditions, for $(X_n)_n$ satisfying (DoA) and assuming the bias condition:

$$\sqrt{n/r}\Big(\mathbb{E}[h(Z_{r,1}, Z_{r,2})] - \mathbb{E}[h(Z_1, Z_2)]\Big) \to B,$$

we have

$$\sqrt{n/r} \cdot \left\{ \frac{U_n^{\mathrm{mb}}}{f(a_r)} - \vartheta \right\} \rightsquigarrow \mathcal{N}(B, \sigma^2_{\mathrm{mb}}),$$

where $Z_{r,i} = (M_{r,i} - b_r)/a_r$; $Z_1, Z_2 \sim \mathrm{GEV}(\gamma)$ i.i.d. and $\vartheta = \mathbb{E}[h(Z_1, Z_2)]$.
Again, it holds that $\sigma^2_{db} \geq \sigma^2_{sb}$.

Probability weighted moment estimator

1. Let $M \sim G_\eta, \eta = (\mu, \sigma, \gamma)$.
2. For $k \in \mathbb{N}_0$ the probability weighted moment is given by

$$\beta_{\eta,k} = \mathbb{E}[MG_\eta^k(M)] = \frac{1}{k+1}\Big[\mu - \frac{\sigma}{\gamma}\{1 - (k+1)^\gamma \Gamma(1-\gamma)\}\Big].$$

3. Well known: There exists a differentiable one-to-one function H with $H(\beta_0, \beta_1, \beta_2) = \eta$.
4. Well known estimator with sample $\mathcal{M} = \{M_1, \ldots, M_n\}$ given by

$$\hat{\beta}_0(\mathcal{M}) = \frac{1}{n}\sum_{i=1}^n M_i, \quad \hat{\beta}_1(\mathcal{M}) = \frac{1}{n}\sum_{i=1}^n \frac{i-1}{n-1} M_{(i)}, \quad \hat{\beta}_2(\mathcal{M}) = \frac{1}{n}\sum_{i=1}^n \frac{(i-1)(i-2)}{(n-1)(n-2)M_{(i)}}.$$

5. If there are no ties with probability 1, then identification as U-statistic with kernel $\max M_1, \ldots, M_k/k$ is possible.

Piecewise stationarity

- **Idea**: Independence between seasons but stationarity inside
- Think of air temperatures (detrending), precipitation

$$
\begin{aligned}
&(X_1, \ldots, X_r, X_{r+1}, \ldots, X_{2r}, \ldots\ldots, X_{(m-1)\cdot r+1}, \ldots, X_{m\cdot r}) \\
&= (Y_1^{(1)}, \ldots, Y_r^{(1)}, Y_1^{(2)}, \ldots, Y_r^{(2)}, \ldots, Y_1^{(m)}, \ldots, Y_r^{(m)}),
\end{aligned}
$$

where $(Y_j^{(t)})_j$ are i.i.d. copies of a strictly stationary time series satisfying (DoA).

Consequences:

- $\mathcal{M}_r^{\mathrm{db}}$ is i.i.d.
- $\mathcal{M}_r^{\mathrm{sb}}$ is not stationary anymore. $\rightarrow$ New bias component.

How to apply Block Maxima modelling

Exemplary situation:

- Interested in 100-year return level for the yearly maximal precipitation: $\mathrm{RL}(100, 365)$
- $\mathrm{RL}(100, 365) := F_{365}^{-1}(M_{365})(0.99)$, where the latter is the 0.99-quantile of yearly maximal precipitation distribution.
- Observe daily precipitation P_i (assume stationarity)
- Yearly maximal precipitation $M_s = \max(P_1^{(s)}, \ldots, P_{365}^{(s)})$
- Sensible assumption: $M_s \sim \mathrm{GEV}(\mu, \sigma, \gamma)$.
- Estimate $\eta = (\mu, \sigma, \gamma)$ by means of MLE or PWM.
- Estimator of return level given by $\widehat{\mathrm{RL}}(100, 365) = \mathrm{GEV}(\hat{\eta})^{-1}(0.99)$.

Common air temperature model in climatology

Linear model for the location parameter

$$M_t := \max(T_1^{(t)}, \ldots, T_r^{(t)}) = cx_t + Z_t,$$

- $T_i^{(t)}$ ith daily temperature in the tth season.
- $(Z_t)_t$ is stationary with $Z_t \sim \mathrm{GEV}(\mu, \sigma, \gamma)$.
- $(x_t)_t$ is the sequence of smoothed global mean surface temperatures (observable).
- $\mu, c, \gamma \in \mathbb{R}, \sigma > 0$ are unknown.

1. Estimate c by $\hat{c}$ by robust regression.
2. Detrend M_t to obtain $\hat{Z}_t = M_t - \hat{c}x_t$.
3. Use methodology for stationary GEV models on $(\hat{Z}_t)_t$ “sample”.

Conditions for bootstrap

1. Same conditions as for U-statistics regarding short-range dependence, but $(n/r)^{1+\omega}\alpha(r) = o(1)$ instead of β.
2. Integrability conditions 4.10.1

Asymptotic linearization I

- Problem: $M_{r,i}$ does not converge but $Z_{r,i} = (M_{r,i} - b_r)/a_r$ does.
- Let $\hat{\theta}_n = \varphi_n(M_{r,1}, \dots, M_{r,n})$ for $\varphi_n \colon \mathbb{R}^n \to \mathbb{R}$. Often there exist functions $\psi_n \colon \mathbb{R}^n \to \mathbb{R}$, $A \colon (0, \infty) \times \mathbb{R} \to \mathbb{R}$, $B \colon \mathbb{R} \times (0, \infty) \to \mathbb{R}$ s.t.

$$\varphi(m_1, \dots, m_n) = A(a, b)\psi_n(\frac{m_1 - b}{a}, \dots, \frac{m_n - b}{a}) + B(a, b),$$

  where $m_i, b \in \mathbb{R}, a > 0$.
- Often there exists a sequence ϑ_r solving

$$\theta_r = A(a_r, b_r)\vartheta_r + B(a_r, b_r).$$

- Then, with $\hat{\vartheta}_n = \psi_n(Z_{r,1}, \dots, Z_{r,n})$

$$\hat{\theta}_n - \theta_r = A(a_r, b_r)(\hat{\vartheta}_n - \vartheta_r).$$

- Often (for instance, by the delta-method) one can linearize

$$\sqrt{\frac{n}{r}}(\hat{\vartheta}_n - \vartheta_r) = \bar{\mathbb{G}}_{n,r}^{\mathrm{mb}}(h_n) + o_{\mathbb{P}}(1).$$

Asymptotic linearization II

- Thus

$$A(a_r, b_r)^{-1}\sqrt{\frac{n}{r}}(\hat{\theta}_n - \theta_r) = \mathcal{N}(0, \sigma^2_{\mathrm{mb}}) + o_{\mathbb{P}}(1).$$

Example:

1. Consider again variance of block maximum σ_r^2 with estimator $\hat{\theta}_n^{\mathrm{mb}} = U_n^{\mathrm{mb}}(h_{\mathrm{Var}})$
2. $\hat{\theta}_n^{\mathrm{mb}} - \theta_r = a_r^{-2}(\hat{\vartheta}_r - \vartheta_r)$
3. The right side without a_r^{-2} has by theory of U-statistics of Block Maxima the linearization

$$\sqrt{\frac{n}{r}}\frac{2}{n_{\mathrm{mb}}}\sum_{i=1}^{n_{\mathrm{mb}}} h_{1,r}(Z_{r,i}^{\mathrm{mb}}) + o_{\mathbb{P}}(1).$$

Bootstrap consistency: Proof sketch

- By linearization condition: Only need to consider the rescaled maxima.
- Denote the empirical processes

$$\mathbb{G}_n^{\mathrm{cb}} = \mathbb{P}_n^{\mathrm{cb}} - P_n^{\mathrm{cb}}, \quad \widetilde{\mathbb{G}}_n^{\mathrm{cb}} = \mathbb{P}_n^{\mathrm{cb}} - P_r,$$

  where $\mathbb{P}_n^{\mathrm{cb}} = \frac{1}{n}\sum_{i=1}^n \delta_{Z_{r,i}^{\mathrm{cb}}}$, $P_n^{\mathrm{cb}} = \frac{1}{n}\sum_{i=1}^n \mathbb{P}(Z_{r,i}^{\mathrm{cb}} \in \cdot\,)$, $P_r = \mathbb{P}(Z_{r,1} \in \cdot\,)$.
- Denote the empirical measure of the rescaled bootstrap resample

$$\widehat{\mathbb{P}}_n^{\mathrm{cb},*} = n^{-1}\sum_{i=1}^n \delta_{Z_{r,i}^{\mathrm{cb},*}} = \frac{1}{n}\sum_{i=1}^{m(2)} W_{n,i} \sum_{s \in I_{2r,i}} \delta_{Z_{r,s}^{\mathrm{cb}}},$$

  where $m(2) = n/(2r)$, $\boldsymbol{W}_n$ is multinomial with $m(2)$ trials and class probabilities $1/m(2)$; and $I_{2r,i}$ is the ith block of $2r$ indices. Denote the associated empirical process as

$$\widehat{\mathbb{G}}_n^{\mathrm{cb},*} = \sqrt{\frac{n}{r}}\left(\widehat{\mathbb{P}}_n^{\mathrm{cb},*} - P_n^{\mathrm{cb}}\right).$$

Proof continued

1. Establish clt for circmax estimator $\mathbb{G}_n^{\mathrm{cb}}$. (Similar to U-statistics clt)
2. Use following result from Bücher and Kojadinovic (2019).

Theorem: For statistics $S_n \rightsquigarrow Q$ and bootstrap replicates $S_n^{(1)}, S_n^{(2)}$:

$$(S_n, S_n^{(1)}, S_n^{(2)}) \rightsquigarrow Q^{\otimes 3} \Leftrightarrow d_K\left(\mathcal{L}(S_n), \mathcal{L}(S_n^{(1)}|X_1, \dots, X_n)\right) = o_{\mathbb{P}}(1).$$

3. Use "Poissonization:" If $N_n \sim \mathrm{Poi}(m(2))$ independent of $\boldsymbol{W}_n = \sum_{j=1}^{m(2)} \boldsymbol{U}_{n,j}$, where $(\boldsymbol{U}_{n,j})_j$ are i.i.d. multinomial with 1 trial and $m(2)$ classes with the same probability $m(2)$. Then $\widetilde{\boldsymbol{W}}_n = \sum_{j=1}^{N_n} \boldsymbol{U}_{n,j} \sim \mathrm{Poi}^{\otimes m(2)}$.
4. Decompose Cramér-Wold expression into (independent) multiplier bootstrap via last construction and error term of removing dependency.