Exceptional event analysis in the deep tail
or
Consistency of estimates of parameters in models with fixed covariates

Axel Bücher, Johan Segers, **Torben Staud**

No Pro(o)f-Seminar
April 1, 2025

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Motivation

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Problem

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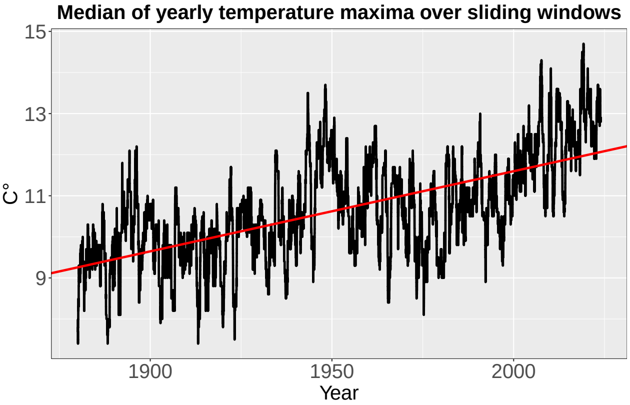
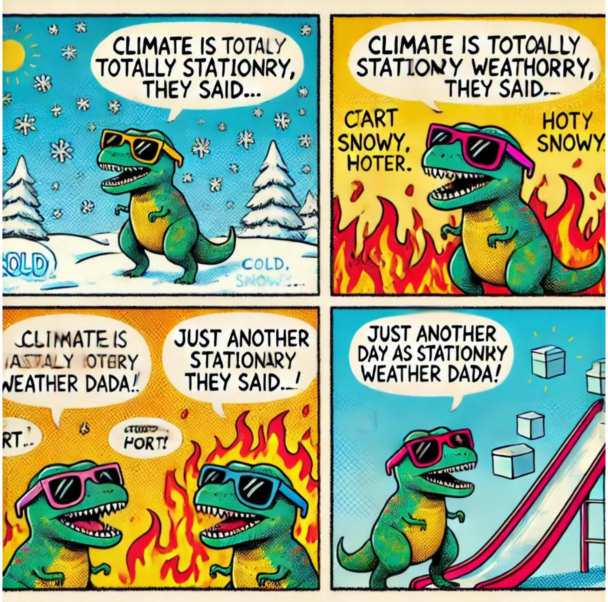


T-Rex says


CLIMATE IS TOTALY TOTALLY STATIONRY, THEY SAID...
COLD.
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State of the Art climate models

Temperature Events

- Trend modelled as *shift fit* with GMST
- $M_t = \mathrm{GEV}(\mu_t, \sigma_0, \xi_0)$

$$\mu_t = \mu_0 + \alpha_0 \times \mathrm{GMST}(t).$$

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Combination (*Shift and Scale fit*) also possible (not yet standard though)

Our Solution

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Tackle non-stationarity by treating it as a covariate (fixed setting)

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$$Y_{n,i} \sim Q_n(x_{n,i}, \mathrm{d}y),$$

where $Q_n : \mathbb{X} \times \mathcal{B}(\mathbb{Y}) \to [0,1]$ is a *regular* sequence of Markov-Kernels.

Think of $Y_{n,t}$ = Maximal temperature of the $t-$th year (depends on t via the covariate $\mathrm{GMST}(t)$).
Toy example:

$$Y_{n,i} \sim \frac{i}{n} + \mathcal{N}(1/n, 1).$$

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- Criterion functions:

$$M_n(\eta) = \mathbb{P}_n m_\eta = \frac{1}{n}\sum_{i=1}^{n} m_\eta(x_{n,i}, Y_{n,i}) \qquad \text{(subasymptotic)}$$

$$M(\eta) = \int_{\mathbb{X}}\int_{\mathbb{Y}} m_\eta(x, y)\, Q(x, \mathrm{d}y) P_X(\mathrm{d}x) \qquad \text{(tentative limit).}$$

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Now (kinda) maximize the arg of $M_n(\cdot)$ to estimate the argmax $\eta_0 \in H$ of $M(\cdot)$!

Considered estimators

All estimators which satisfy this *asymptotic maximizer condition*

$$M_n(\hat{\eta}_n) \geq M_n(\arg\max_{\tilde{\eta} \in H} M(\tilde{\eta})) - o_{a.s.}(1).$$

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Essence: $\hat{\eta}_n$ should asymptotically maximize $M_n(\cdot)$.

Result

Theorem Axel, Johan, Torben (2024+): Under *some* conditions we have that $\hat{\eta}_n$ consistently estimates η_0 in the following sense

$$\mathbb{P}\big[\lim_{n\to\infty} \hat{\eta}_n = \eta_0\big] = 1,$$

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That's a very general result:

- Dependence on covariate allowed via x.
- Non-stationarity of the $Y_{n,i}$ possible via dependence on the Feller-Markov kernels Q_n and $x_{n,i}$.
- Even non regular models like "support changes with η" (think of GEV model or uniform model $X \sim \mathcal{U}(0,\eta)$) allowed, since $m = -\infty$ allowed.
- We can handle even more general estimators than M-estimators.

Applications

Conditional MLE

- Model:

$$Y_{n,i} \sim P_{\theta(x_{n,i},\eta_0)},$$

- $(P_\theta : \theta)$ is a parametric family (think of GEV shift/scale fit) with densities wrt to a $\sigma-$finite measure ν.
- η_0 is unknown.
- $\theta \colon H \to \Theta$ is a known link function (like the scale/shift fit functions).
- We observe $Y_{n,i}$ and the covariates $x_{n,i}$.
- $Q_n \equiv Q = P_{\theta(\cdot,\eta_0)}$ is constant in n in this example.

Pairwise likelihood from multivariate extremes

- Now $\mathbb{Y} \subset \mathbb{R}^d, Y_{n,i} \sim P_{\theta(x_{n,i},\eta_0)}$ as before
- Sometimes calulcating the d-variate density is expensive. Solution:
- *Pairwise criterion function*

$$m_\eta(x,y) = \sum_{(j,l)\in D_2} \log p_{\theta(x,\eta)}(j,l),$$

where $D_2 = \{(j,l)\colon 1 \leq j \neq l \leq d\}$ and $p_\theta^{(j,l)}$ denotes the bivariate marginal (of d) density corresponding to j,l coordinates.

Optimum score estimation

- Fancy Data science
- Find a *scoring rule* S under which one calibrates a model.
- S takes two arguments P, y where P is a probability measure and y an observation. $S(P, Q) := \int S(P, y)\,\mathrm{d}Q(y)$.
- Identifiability of the model corresponds to assuming $S(P, P) > S(P, Q)$ for all $Q \neq P$ (unique maximum).
- We then have the estimator

$$\hat{\eta}_n = \arg\max_{\eta \in H} \frac{1}{n} \sum_{i=1}^{n} S(P_{\theta(x_{n,i}, \eta)}, y_i).$$

And quite some more examples (functional data . . .).

Outlook

- We have consistency (minimal property).
- Asymptotic normality next step (want confidence intervals and tests).
- Bootstraps?
- Dependence between the $Y_{n,i}$ (with mixing this shouldn't be to much of a problem).

Most important references

Bücher, A., Segers, J. and Staud., T. Consistency of estimates of parameters in models with fixed covariates (**WiP**).

Dombry, C. (2015). Existence and consistency of the maximum likelihood estimators for the extreme value index within the block maxima framework. *Bernoulli*, Vol. 21, No. 1

Philip S., et al. (2020). A protocol for probabilistic extreme event attribution analyses. *Adv. Stat. Clim. Meteorol. Oceanogr.*, 6, 177-203.

Thank you

Conditions

Weak convergence regularities

1. There exists a non-degenerate RV X on $\mathbb{X}$ such that

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{x_{n,i}} \rightsquigarrow P_X;$$

think of the uniform design in regression.

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2. There exists a Markov kernel $Q\colon \mathbb{X}\times\mathcal{B}(\mathbb{Y})\to[0,1]$ such that $x\mapsto Q(x,\,\mathrm{d}y)$ is weakly continuous and such that $(Q_n(\cdot,\,\mathrm{d}y)_n)$ converges continuously to $Q(\cdot,\,\mathrm{d}y)$ in the weak topology.

Identifiability

The asymptotic criterion function M has a unique point of maximum η_0, that is for $\eta \neq \eta_0$

$$M(\eta) = \int_{\mathbb{X}} \int_{\mathbb{Y}} m_\eta(x, y)\, Q(x,\, \mathrm{d}y) P_X(\,\mathrm{d}x) < \int_{\mathbb{X}} \int_{\mathbb{Y}} m_{\eta_0}(x, y)\, Q(x,\, \mathrm{d}y) P_X(\,\mathrm{d}x) = M(\eta_0).$$

Boundedness

1. The criterion function is bounded from above (may be unbounded below)
2. Q_n is L^2 sochastically dominated, that is, there exists $W \in L^2$ with

$$\forall t \geq 0, n \in \mathbb{N}, x \in \mathbb{X} \colon Q_n\big(x, \{y \in \mathbb{Y} \colon |m_{\eta_0}(x,y)| > t\}\big) \leq P(W > t).$$

Space and criterion function regularity

1. The metric spaces $(\mathbb{X}, d_{\mathbb{X}})$ and (H, d_H) are compact
2. The criterion function m is upper semicontinuous.
3. The function $(x, y) \mapsto m_{\eta_0}(x, y)$ is continuous $P_{X,Y}$ a.e.